Scaling and Estimation

Physics 281

Lecture 1
- Buckingham Pi Theorem
  - ⇒ Systematics of Dimensional Analysis.
- * – context: Drag

→ Here consider systematics of dimensional analysis; scaling arguments.

→ Why scaling?

- real problems – nonlinear
  - many degrees freedom
- need estimation, guides to computation
  - n.b. {computers are poor at asymptotics / but asymptotics fundamental}

→ Common issues:

describe:

– self-similarity → concept in problems where phenomena "look the same", over broad range of scales

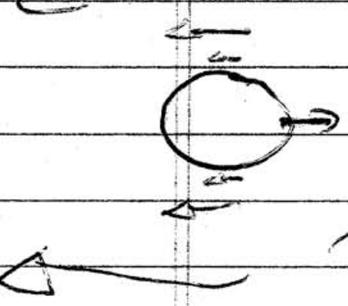
e.g. {blast wave / turbulence}

⇒ scaling relations constitute "the answer"

→ emergent scale ⇒ i.e. nonlinear dynamics defines "new scale"; e.g. Rhines scale, i.e. usually lin ~ nonlin balance.

→ new physics ... (it happens!)

→ boundary layer ⇒ transition between different scaling regimes



no slip b.c. $V_T\big|_{surf} = 0$

~ potential flow far from body

~ boundary layer near body, viscosity important

⇒ different scalings, which must be matched.

⇒ Basic Question:

→ How Formulate Systematics of Dimensional Analysis? Limitations?

→ When might it fail?

→ discuss in context of familiar example:

Drag on Sphere (body) in Fluid ? Body

(a) → V ρ F_d ?

Preliminaries

→ how is drag quantified / i.e. score kept ?

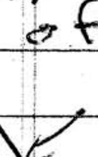
$$F_d \sim C_D\, S_A\, V^2 \rho$$

C_D: drag coefficient; V: speed; S_A: surface area

$C_D \equiv$ drag "coefficient"

$C_D = C_D(Re, Ma, shape, \ldots)$

"idea" of C_D:



$$\Delta P \sim \rho V^2 A (\Delta t)$$

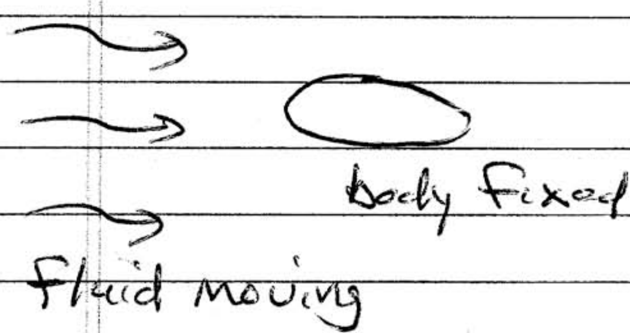
ΔP: change in momentum of ball; ρV^2: Pressure ~ force/area; Δt: time increment

$$\sim (\rho V A)(V \Delta t)$$

but $l_{max} \ll V\Delta t$

→ What, *really*, is drag?

Consider frame where:



Drag ≡ rate of *removal* of momentum from moving fluid by immersed body

'Removal' → transmission in boundary, surface layer
→ no-slip condition / viscosity.

leaves *wake* ⇒ water acts to fill in dead region

$\Delta p/\Delta t \sim F_d \sim \rho A V^2$
$\sim C_D \rho A V^2$

C_D captures additional physics.

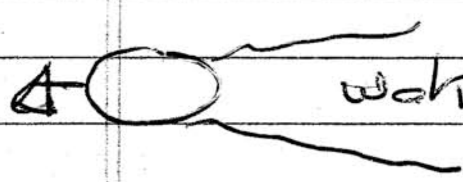
Why additional physics? → see 3a

(Why not formula valid?)

⇒

- drag ⟺ wake ⇒

ahead - behind asymmetry



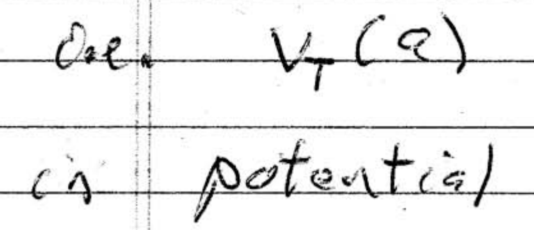
wake → irreversibility
see 4a, contrast

- wake is consequence of no-slip b.c.

i.e. $V_T(a) = 0$

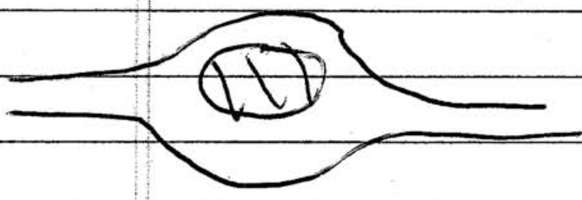
- in potential flow, $\nabla^2\phi = 0$, $\underline{v} = \nabla\phi$

V_T can be large ⟺ no wake



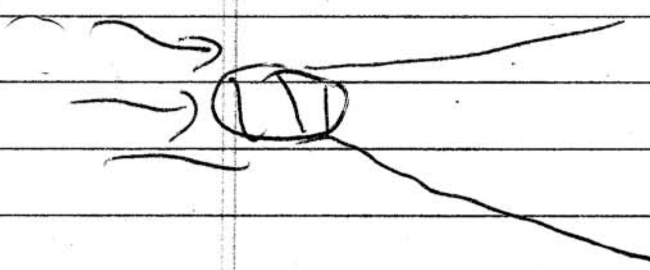
$\Delta V_{\perp 1} = \Delta V_{\perp 2}$
$\Delta V_{trans.}$ deflection
same upstream, downstream.

ideal Fluid — Euler eqn. reversible



$\leftarrow$ vs $\rightarrow$ reversible

Viscous fluid — N-S Eqn. irreversible



$\leftarrow$ vs $\rightarrow$ clearly different

— wake, and thus drag, are due to

→ viscosity
irreversibility ↓
→ vorticity

} highlight role of B.L. around object.

n.b. highlights role of boundary layers ⟺ drag can be independent of Re, but viscosity needed to satisfy B.C.

— additional physics → model

Model → Navier-Stokes Equation $\underline{\nabla} \cdot \underline{v} = 0$

kinematic viscosity

$$\rho \left(\partial_t \underline{v} + \underset{①}{\underline{v} \cdot \underline{\nabla}\, \underline{v}} - \underset{②}{\nu \nabla^2 \underline{v}} \right) = -\underline{\nabla} P$$

advection nonlinear

linear 2nd order [$\nu = 0$ is singular]

Compressible NS Eqn:

$$\rho \frac{d\underline{V}}{dt} = -\underline{\nabla} p + \rho \nu \nabla^2 \underline{V} + (\eta + \mu)\nabla(\nabla \cdot \underline{V})$$

$s = \text{const}, \quad ds = 0$

$\therefore \dfrac{dP}{\rho} = V\,dp - \cancel{T\,ds} = dw$

enthalpy

mostly

- $\nabla \cdot \underline{V} = 0 \rightarrow$ determines pressure
- $\nabla \cdot \underline{V} \neq 0 \rightarrow$ need eqn. state.

Key dimensionless Parameters:

$①/② \equiv$ Reynolds #

$\cong Re \sim VL/\nu$

Models (i.e. eqns) are useful to identify dim-less parameters

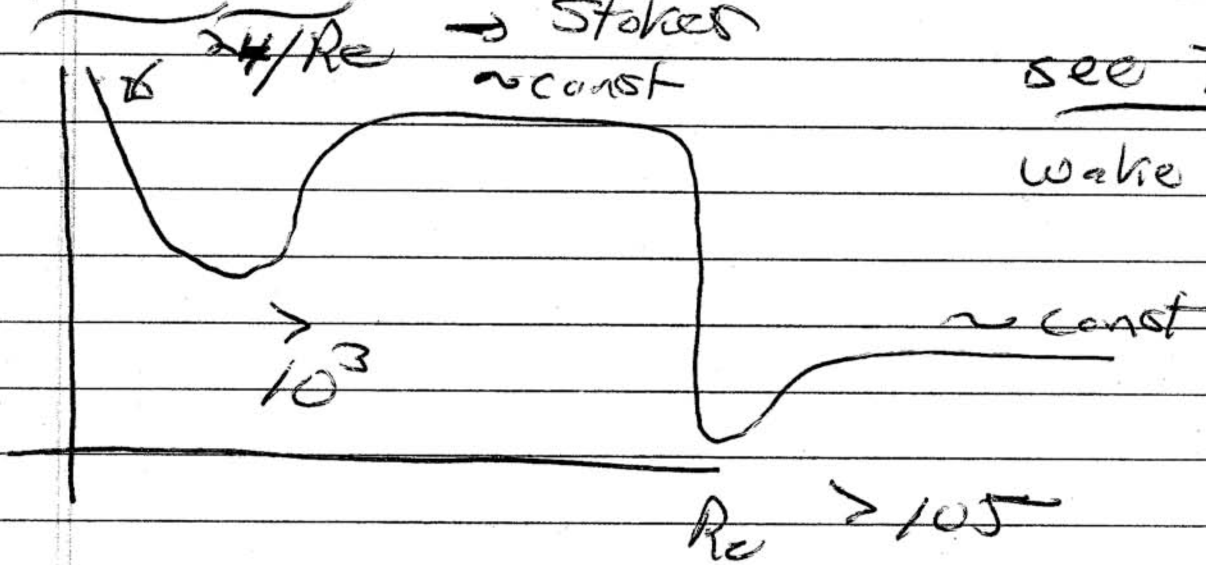
expresses ratio of nonlinear, inertial term to linear, diffusive term.

$Re < 1 \rightarrow$ viscous flow — Stokes

$Re > 1 \rightarrow$ laminar flow — Blasius

$Re > 10^5 - 10^6 \rightarrow$ turbulent flow.

→ the facts:



see 7a.
wake str.

See especially: → basic trends

→ small Re regime → Stokesian regime

$C_D \sim 1/Re$ $\boxed{F_d \sim V}$

i.e. $\underbrace{\cancel{\partial_t \underline{v}}}_{st} + \underbrace{\cancel{\underline{v}\cdot\nabla\underline{v}}}_{Re \ll 1} - \nu\nabla^2\underline{v} = -\frac{\nabla p}{\rho}$

→ large Re → turbulent

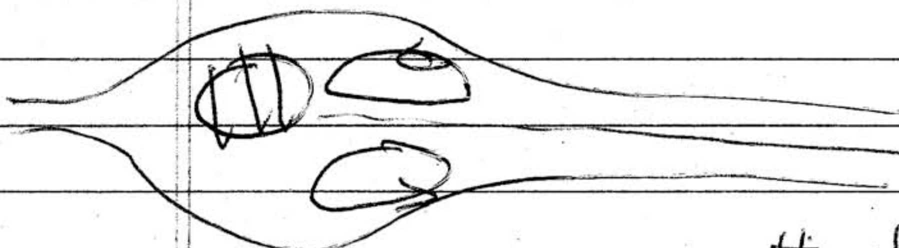
$C_D \sim Re^{(0)}$ → notable, why?

How recover scalings?

Wake structure: (cyl.)

$Re < 10$

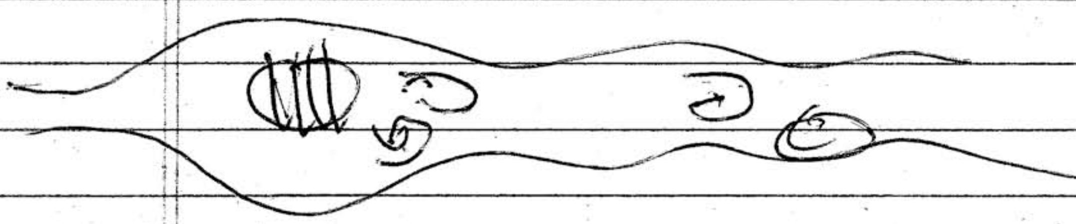
creeping flow



$10 < Re < 40$

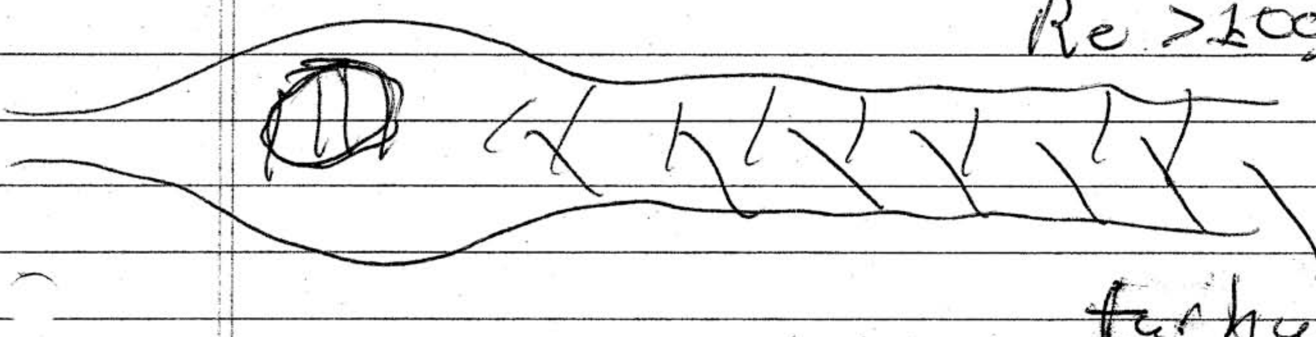
attached vortices

$40 < Re < 200{,}000$



vortex trail

$Re > 200{,}00$



turbulent wake

⇒ Buckingham's Pi Theorem

- e.x. first

- how do dimensional analysis systematically ?

① - identify physically relevant variables

i.e. n of these

→ required answer should depend on relevant variables, and be listed among.

So, for drag on sphere: steady

dim

$R \rightarrow$ length L

$V \rightarrow$ velocity L/T

$\nu \rightarrow$ viscosity (diffusion) L^2/T

$\rho_0 \rightarrow$ fluid density M/L^3

temp.

$\nu \sim v_{th} \, l_{mfp}$

lower in water

water → $10^{-6} \, m^2/sec$

air → $10^{-5} \, m^2/sec$

$F_d \to$ drag force $\quad F_d \sim ML^2/T^2$

② Count indep. dimensions.

$\Rightarrow$ M, L, T

so $n = 5$ quantities

$r = 3$ indep. dims.

$\Rightarrow$

③ $n - r = 2$ dimensionless ratios possible

1 involves F_d.

$\Rightarrow$ $\boxed{\Pi \text{ theorem}}$:

$n \to$ indep. quantities

$r \to$ indep. dimensions

$\therefore$ $n - r$ dimensionless ratios s/t

$$\Pi_i = f(\Pi_1, \Pi_2, \ldots \Pi_{i-1}, \Pi_{i+1}, \ldots \Pi_{n-r})$$

i.e. can express dimensionless ratios in terms each other

Practical Matters

→ one Π involves answer

→ other Π's involve relevant dim-less parameters in model.

n.b. here — model → useful.

→ use insight # values (i.e. experiment) to eliminate some Π variables

i.e. $\ll 1$, or $\gg 1$.

back to sphere example:

→ 2 Π variables.

where $\Pi_1 = f(\Pi_2)$

i.e. $\Pi_1 = F_d / \rho_o R^2 V^2$ → involves answer

$\Pi_2 = Re = VR/\nu$ → suggested as natural dim/less ratio from model

and Π_i theorem ⇒

$$\Pi_1 = \Pi_1(\Pi_2) = f(\Pi_2)$$

⇒ $$\boxed{F_d = f(Re)\,\rho_o R^2 V^2}$$ } Π theorem result for drag.

- basic result
- plausible that drag depends on Re
- not a unique answer

Now, key step:

- explore asymptotic behavior, i.e.

$Re \ll 1$, $Re \gg 1$?

→ $Re \ll 1$

- have $F_d \sim \rho_0 R^2 V^2 f(Re)$
- $Re \ll 1$ → expect $F_d \sim \nu$

  i.e. proportional to viscosity

so

$$f(Re) \sim 1/Re \quad , \quad \text{for } Re < 1.$$

$$F_d \sim \rho R^2 V^2 \frac{\nu}{RV} \sim [\rho R \nu] \, V$$

$$\boxed{F_d \sim [\rho \nu R] \, V \sim \eta R V}$$

- corresponds to Stokesian flow drag

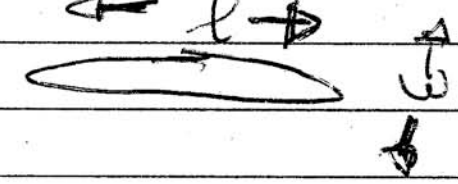
- general result:

EOM : $\quad -\nu \nabla^2 \underline{v} = -\frac{\underline{\nabla} p}{\rho}$

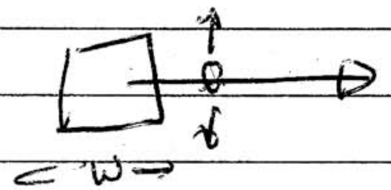
longest dim.

$$F_d \sim 6\pi \eta \, l \, V$$

$$\eta = \rho \nu$$



— Consider flat plate moving head on:

i.e. 

$Re \ll 1$.

Show Stokes scaling applies!

$$F_d \sim PA \sim P \underbrace{l w}$$

for P

$$\nu \nabla^2 V = -\frac{\nabla P}{\rho}$$

what is the scale? → smaller will dominate gradients

⇒

$$\frac{P}{\rho w} \sim \frac{\nu}{w^2} V$$

so

$$P \sim \frac{\rho \nu}{w} V$$

$$F_d \sim PA \sim \frac{\rho \nu}{\cancel{w}} V \quad (\cancel{w} \ell)$$

$$\sim [\rho \nu \ell] V$$

Show for edge-on incidence — HW!

Now, for $Re \gg 1$, expect,

$$F_d \sim \nu^{(0)}$$

but

$$F_d \sim \rho V^2 R^2 f(Re)$$

$$\sim \rho V^2 R^2 Re^{(0)} \to \rho V^2 R^2$$

so $Re \to \infty$

$$F_d \sim \rho V^2 R^2$$

Note:

- $f(Re) \sim Re^{\alpha}$ — unsure about small corrections to α

  $\alpha \ll 1$ is possible

- amazing that $F_d \sim Re^{(0)}$ at large Re ?!

  No ν dependence!

  yet drag requires viscosity !?

  Why?

  → on small scales in BL, ν matters

  i.e. $Re = Re[\text{scale}]$

  $$= \frac{v\ell}{\nu}$$

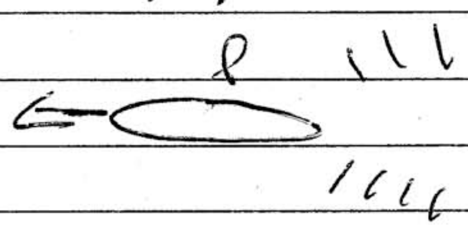
  → but momentum transport to small scales independent of ν (i.e. inertial).

→ Ex. 2

What are the scaling rules for motion of surface ship of length l, speed V? Consider:

i.e. Drag scaling? – assume due surface waves
Model size dependence?
stam/ine

Physics:



with g.

surface ship – modest speed
10–15 m/sec.
– radiates waves.

so drag mechanism is radiative
i.e. akin radiation-induced slowing down of charge.

Now, with gravity:

$$\partial_t \underline{v} + \underline{v}\cdot\nabla\underline{v} = -\frac{\nabla p}{\rho} - g\hat{z} + \nu\nabla^2\underline{v}$$

Parameters:

$V \rightarrow L/T$

$\ell \rightarrow L$

$\rho_o \rightarrow M/L^3$

$\nu \rightarrow L^2/T$

$F_d \rightarrow ML/T^2$

and

$g \rightarrow L/T^2$ $\underline{n=6}$

indep: $\underline{r=3}$ M, L, T

$\Rightarrow \underline{\underline{3}}$ Π variables

as before:

$\Pi_1 = F_d/\rho_o \ell^2 V^2$

$\Pi_2 = V\ell/\nu \equiv Re$

Need one more dim-less ratio

now: – obviously must involve g.

– expect $Re \ggg 1$.

$$\overset{①}{\partial_t V + \underline{V} \cdot \underline{\nabla} \underline{V}} = -\frac{\underline{\nabla} p}{\rho} \overset{②}{- g \hat{z}} + \nu \nabla^2 \underline{V}$$

$\Rightarrow$

$$\Pi_3 \sim ① / ②$$

$$\sim V^2 / Lg \qquad \text{Froude \#} \; Fr$$

$$\sim V^2 / \ell g = Fr$$

$\therefore$ now

$$\Pi_1 = \Pi_1 (\Pi_2, \Pi_3)$$

$\Rightarrow$

$$F_d / \rho \ell^2 V^2 \cong f(Re, Fr)$$

$$\cong f\left(\frac{V\ell}{\nu}, \frac{V^2}{\ell g}\right)$$

Now:

— $Re \to \infty$, F_d indep Re

— $F_d \cong \rho l^2 V^2 f\left(V^2/lg\right)$

$\Rightarrow$ expect F_d must <u>increase</u> with g

<u>i.e.</u> wave drag increases with g.

$\therefore$

$$F_d \cong \rho l^2 V^2 \left(V^2/lg\right)^{\alpha}$$

so $\alpha \cong -1$

$$F_d = \rho l^2 \cancel{V^2} \left(\frac{lg}{\cancel{V^2}}\right)$$

$$\boxed{F_d \sim \rho l^3 g}$$

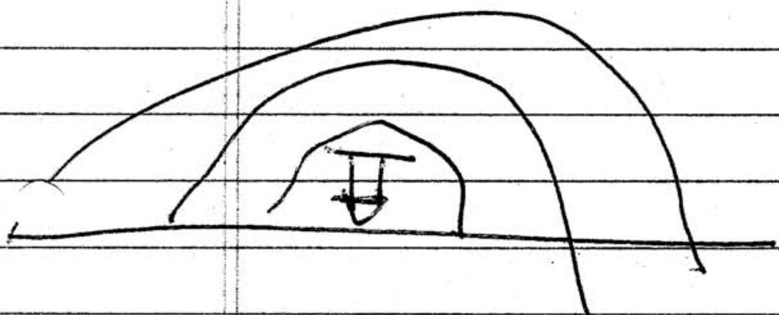
so $F_d \sim \rho l^3 g$

and scaling $\sim (\text{size})^3$.

→ Blast Wave

$\begin{cases} \text{Bomb} \\ \text{SNR} \\ \vdots \end{cases}$

Sedov – Taylor Blast Wave



$r_0 < r < r_{max}$

↑ inner scale

↑ outer scale

blast wave "looks the same" between inner, outer structure

⇒ self-similar structure

– sudden release of energy $E_0 \gg r_0^3 P_0$ at $t = t_0$

– $E \gg P_{amb} R^3$

– blast expands as hemisphere until

– runs out at R_{max}

$E / R_{max}^3 \sim P_{amb}$

For spherical blast, can write eqns of adiabatic gas dynamics as:

$$\partial_t V_r + V_r \partial_r V_r = -\partial_r P/\rho$$

$$\partial_t \rho + \frac{1}{r^2}(r^2 \rho V_r) = 0$$

$$\partial_t (P/\rho^\gamma) + V_r \partial_r (P/\rho^\gamma) = 0$$

seek $\rho(r,t)$, but self-similarity $\Rightarrow$

$$\rho(r,t) = \rho_0(t)\, \rho(r/R(t))$$

i.e. solutions look same at diff'nt times

$\sim$ blast radius $\Rightarrow$ time prog is re-scaling

→ spatial structure self-similar, relative to expanding radius.

→ structure re-scales in time, preserving shape.

Simple example: wave eqn

$$\partial_t^2 u = c^2 \partial_x^2 u \quad \text{(1D linear wave)}$$

$$u = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

i.e. x, t enter as $x \pm ct$, only.

Now, Π-thm. approach: What is radius, in time?

$R \to$ radius $\quad L$

$V \to$ speed of front $\quad L/T$

$\rho \to$ density $\quad M/L^3$

$p \to$ pressure $\quad ML^2/T^2L^3$

$E \to$ energy $\quad ML^2/T^2$

$n = 5$
$r = 3$ $\Rightarrow 2\ \Pi$'s.

$\Pi_1 = R/Vt \quad \to$ unknown ans.

$\Pi_2 = E/\rho R^3 V^2 \quad \to$ dimless ratio for energy so

$$\Pi_1 = f(\Pi_2)$$

i.e.

$$\frac{R}{Vt} = f\left(E/\rho V^2 R^3\right)$$

here

- $f = \text{const} \sim 1$, so

$$R \sim Vt$$

- $f = \text{const}$ and/or energy balance

$$E \sim \rho V^2 R^3$$

so

$$V \sim \left[E/\rho R^3\right]^{1/2}$$

and

$$R \sim \left[E/\rho R^3\right]^{1/2} t$$

$$R^{5/2} \sim (E/\rho)^{1/2}\, t$$

so $$\boxed{R \sim \left(E_0 t^2/\rho_0\right)^{1/5}}$$

Sedov - Taylor Blast wave radius.

N.B.:

- could just put:

$$E \sim \rho R^3 V^2 \sim \rho R^3 \frac{R^2}{t^2}$$

- $$p \sim \rho V^2 \sim \rho_0 R^2/t^2$$
$$\sim \left(E/\rho_0\right)^{2/5} \rho_0 t^{-6/5}$$

$\downarrow$

dynamic pressure drops with time.